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Abstract Schroedinger-Type Differential Equations with Variable Domain*

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The Cauchy problem is studied for a class of linear abstract differential equations of Schroedinger-type with variable domain. Existence and uniqueness results are proved for (suitably defined) weak solutions. Some applications are given: they concern initial-boundary value problems for linear Schroedinger-type P.D.E., either with mixed variable lateral conditions or in non-cylindrical regions.

1. INTRODUCTION

We study, in this paper, the following abstract Cauchy problem,

$$u(t) \in V(t), \qquad \text{for a.a. } t \in]0, T[, \tag{1.1}$$

$$u'(t) + iA(t)u(t) = f(t), \qquad \text{in }]0, T[, \tag{1.2}$$

$$u(0) = u_0, \tag{1.3}$$

where $0 < T < +\infty$; $V \subseteq H \equiv H^* \subseteq V^*$ is the standard complex Hilbert triplet; $\{V(t)\}_{t \in [0,T]}$ is a family of closed subspaces of V, which are *not* dense in H, in general; "$t \to A(t)$" is a given operator-valued function from $]0, T[$ into $\mathcal{L}(V, V^*)$ (such that, for a.a. $t \in]0, T[$, the operator $A(t)$ is hermitian symmetric and weakly V-coercive); "$t \to f(t)$" is a given V^*-valued function, and $u_0 \in V(0)$. Then, we ask for some V-valued function "$t \to u(t)$," which solves (1.1)–(1.2)–(1.3) *in some suitable weak sense*. We

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observe that, through (1.1)–(1.2)–(1.3) (and in the previous setting of variational type), we can deal with several initial-boundary value problems for linear P.D.E. of Schroedinger-type, as, e.g., Cauchy-mixed problems, *with mixed variable lateral conditions*, or Cauchy–Dirichlet problems *in non-cylindrical regions*. We want to remark that such problems could also be handled through other abstract approaches (e.g., the semigroups approach), by only considering (1.2)–(1.3) where, for a.a. $t \in]0, T[$, $A(t)$ is an unbounded (self-adjoint positive) operator in H, and the domain $D(A(t))$ of $A(t)$ (*and also* $D(A^{1/2}(t))$) *can vary with* t. So, by means of such remarks, we can *also* refer to (1.1)–(1.2)–(1.3) as a Cauchy problem for a linear abstract differential equation of Schroedinger-type *with variable domain*.

It is well known that the "usual" Cauchy problem for linear abstract differential equations of Schroedinger-type was studied by several authors: we refer, e.g., to Lions [17], Lions and Magenes [19], Kato [15] (and the references therein), and, *in particular*, to the papers by Pozzi [21, 22]. However, as far as we know, the problem (1.1)–(1.2)–(1.3) (concerning Schroedinger-type equations *with variable domain*) was not studied before our present paper. On the other hand, several results are well known for linear abstract *variable domain* differential equations of *parabolic* or *hyperbolic type*. Considering only the hyperbolic case (which seems to be closer to the case of Schroedinger-type equations), we can refer, e.g., to Da Prato [11], Carroll and State [10], Goldstein [13], Arosio [1], and to our paper [6].

Our present paper is structured as follows. Section 2 concerns, at first, the notation and the main assumptions. Then, we give, through Definition 2.1, a (natural) notion of a *weak* (variational) solution to (1.1)–(1.2)–(1.3). The remaining part of Section 2 concerns various preliminary results: some of them are the same as in [6], so that we have omitted here their proofs (which are given in [6], by also using some results by Savaré [23]). In Section 3 we prove (see Theorem 3.1) that a weak solution to (1.1)–(1.2)–(1.3) *actually exists*, when we assume (besides the other hypotheses on $A(t)$, $f(t)$, u_0) that $\{V(t)\}$ is *a non-decreasing family with* t. Our main tool, in the proof, is a suitable procedure of penalization. Moreover, Theorem 3.2 provides further regularity of the weak solutions, when we *also* assume that the spaces $V(t)$ are dense in H. In Section 4 we prove (see Theorem 4.1) the *uniqueness* of the weak solution to (1.1)–(1.2)–(1.3), when we assume (besides the other hypotheses) that $\{V(t)\}$ is *a non-increasing family with* t. It is clear that, considering our Theorems 3.1 and 4.1 *together*, we can get the existence *and* the uniqueness of the weak solution, *only* in the case where $V(t) \equiv$ *constant*. Anyway, even in the very special case where $V(t) \equiv V$, our results give *something new*, as we observe in Subsection 5.1 of Section 5 below. The remaining part of Section 5 concerns some examples of applications of our abstract results to linear P.D.E. of Schroedinger-type: Subsection 5.2 is devoted to Cauchy-mixed problems, with mixed variable lateral conditions, while Subsection 5.3 concerns Cauchy−Dirichlet problems in non-cylindrical regions.

2. NOTATION AND MAIN ASSUMPTIONS: THE WEAK SOLUTIONS AND SOME PRELIMINARY RESULTS

2.1. Let T be given, with $0 < T < +\infty$. We will use the well known spaces $C^k([0,T]; X)$, $L^p(0,T; X)$, $W^{k,p}(0,T; X)$, where X is some Banach space, $1 \le p \le +\infty$, and k is some non-negative integer. Let now (as, e.g., in Lions and Magenes [19])

$$V \subseteq H \equiv H^* \subseteq V^*, \qquad \text{with } V \text{ separable}, \tag{2.1}$$

be the standard *complex* Hilbert triplet (i.e., V and H are two complex Hilbert spaces, with $V \subseteq H$, and such inclusion is continuous and dense; H is identified with its antidual space H^*, so that H can be continuously and densely imbedded in V^*). $(\cdot,\cdot)$ denotes both the scalar product in H, and the antiduality pairing between V^* and V; $((\cdot,\cdot))$ denotes the scalar product in V. $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$ are respectively the norms in V, H, and V^*. We are also given

$$\text{a family } \{V(t)\}_{t\in[0,T]} \text{ of closed subspaces of } V. \tag{2.2}$$

We will use the following notation. Let p be given, with $1 \le p \le +\infty$; then

$$L^p(0,T;V(t)) \equiv \{w(t) \in L^p(0,T;V) \mid w(t) \in V(t), \text{ for a.a. } t \in]0,T[\}. \tag{2.3}$$

Next, let us introduce a suitable function $A(t)$ of bounded variation on $]0,T[$, with values in the space $\mathsf{L}(V,V^*)$ of linear and continuous operators from V into V^*. So, throughout the present paper, we assume that

$$A(t) \in BV(0,T;\ \mathsf{L}(V,V^*)); \tag{2.4}$$

$$(A(t)u,v) = \overline{(A(t)v,u)},$$

$$\forall u,v \in V, \text{ and for a.a. } t \in]0,T[\ \text{(hermitian symmetry of } A(t)); \tag{2.5}$$

$$\exists c > 0, \text{ and } \exists \lambda \ge 0 \text{ such that}$$

$$(A(t)u,u) + \lambda|u|^2 \ge c\|u\|^2,$$

$$\forall u \in V, \text{ and for a.a. } t \in]0,T[\ \text{(weak } V\text{-coerciveness of } A(t)). \tag{2.6}$$

We will also consider functions $f(t)$, such that

$$f(t) \in BV(0,T;V^*). \tag{2.7}$$

Now, let us introduce a natural notion of weak solution to the problem (1.1)–(1.2)–(1.3). Toward this aim, let us first recall that $W^{1,p}(0,T;X) \subset C^0([0,T];X)$, where $1 \le p \le +\infty$, and X is any Banach space. Then, we define

$$W \equiv \{w(t) \mid w(t) \in L^1(0,T;V(t)) \cap W^{1,1}(0,T;V^*); w(T) = 0\}. \tag{2.8}$$

Next, suppose that $u(t)$ satisfies "formally" (1.1)–(1.2)–(1.3), and "multiply" (in the antiduality pairing between V^* and V) both sides of (1.2) by any $w(t) \in W$; then, integrate from 0 to T. By integrating by parts the term $\int_0^T (u'(t), w(t))\,dt$, and taking into account (1.3), we are led, in a natural way, to give the following definition of weak solution to the problem (1.1)–(1.2)–(1.3).

DEFINITION 2.1. Let (2.1)–(2.2) and (2.4)–(2.5)–(2.6) hold. Let $u_0 \in V(0)$, and $f(t)$ be given, where $f(t)$ satisfies (2.7). Then, we say that $u(t)$ *is a weak solution to the problem* (1.1)–(1.2)–(1.3), *if and only if*

$$u(t) \in L^\infty(0,T;V(t)), \tag{2.9}$$

and the following equality holds:

$$i\int_0^T (A(t)u(t), w(t))\,dt - \int_0^T \overline{(w'(t), u(t))}\,dt$$
$$= \int_0^T (f(t), w(t))\,dt + \overline{(w(0), u_0)},$$
$$\text{for every } w(t) \in W \ (\text{where } W \text{ is defined in } (2.8)). \tag{2.10}$$

We shall prove, in Section 3, an existence result, by assuming (besides the previous hypotheses) that $\{V(t)\}$ is a *non-decreasing* family with t, i.e., that

$$V(t_1) \subseteq V(t_2), \qquad \forall t_1, t_2 \text{ such that } 0 \le t_1 \le t_2 \le T. \tag{2.11}$$

The following subsections are devoted to some preparatory results for the proofs in Sections 3 and 4.

2.2. We assume, throughout this subsection, that (2.1), (2.2), and (2.11) hold. We extend the definition of $\{V(t)\}$ to all of $\mathbb{R}$, by setting

$$V(t) \equiv V(0), \forall t < 0; \qquad V(t) \equiv V(T), \forall t > T. \tag{2.12}$$

Then, $\{V(t)\}_{t\in\mathbb{R}}$ is also a non-decreasing family with t. Next, we define, for every $t \in \mathbb{R}$,

$$\begin{aligned} &\pi(t) \equiv \text{projection operator from } V \text{ onto } V(t); \\ &P(t) \equiv I - \pi(t) \qquad (I = \text{identity operator}). \end{aligned} \tag{2.13}$$

(Hence, $P(t)$ is the projection operator from V onto $V(t)^{\perp}$, where $V(t)^{\perp}$ denotes the orthogonal subspace of $V(t)$ in V. Of course, thanks to the previous assumptions, $\{V(t)^{\perp}\}_{t\in\mathbb{R}}$ is a *non-increasing* family with t.) We remark that the operator function "$\mathbb{R} \ni t \to P(t)$" (with values in $\mathbf{L}(V,V)$) has only, in general, little regularity (even in some "concrete" cases, where the family $\{V(t)\}$ "seems to depend smoothly on t"; see [7]). For our purposes, we need to approximate such a function by more regular maps: we do it by refining an argument used in [5]. So, let $\{\varphi_k(t)\}_{k\geq 1}$ be a sequence of "smoothing kernels," i.e., such that ($\forall k \geq 1$)

$$\begin{aligned} &\varphi_k(t) \in C_0^{\infty}(\mathbb{R}); && \varphi_k(t) \geq 0, \forall t \in \mathbb{R}; \\ &\operatorname{supp}(\varphi_k(t)) \subset [-1/k, 0]; && \int_{-\infty}^{+\infty} \varphi_k(t)\, dt = 1. \end{aligned} \tag{2.14}$$

Then, we define ($\forall k \geq 1$)

$$P_k(t) \equiv P(t) * \varphi_k(t) \equiv \int_{-\infty}^{+\infty} P(t-\tau)\varphi_k(\tau)\, d\tau, \qquad \forall t \in \mathbb{R}. \tag{2.15}$$

It is clear that ($\forall k \geq 1$)

$$\text{“}t \to P_k(t)\text{”} \in C^{\infty}(\mathbb{R}; \mathbf{L}(V,V)). \tag{2.16}$$

Now, we collect some properties of $P_k(t)$ in the following lemma (for its proof we refer to [6]).

LEMMA 2.1. *Let* (2.1), (2.2), (2.11), *and* (2.12) *hold. Let* $P_k(t)$ *be defined through* (2.15) *and* (2.14). *Then, for every* $k \geq 1$, *for any* $t \in \mathbb{R}$, *and any* $u, v \in V$, *it results that*

$$((P_k(t)u, v)) = ((u, P_k(t)v)); \tag{2.17}$$

$$\|P_k(t)v\|^2 \leq ((P_k(t)v, v)); \tag{2.18}$$

$$\|P_k(t)v\| \leq \|P(t)v\| \qquad (\textit{and, of course, } \|P(t)v\| \leq \|v\|); \tag{2.19}$$

$$\frac{d}{dt}((P_k(t)v, v)) \equiv ((P_k'(t)v, v)) \leq 0. \tag{2.20}$$

The following remark will also be important in the sequel.

Remark 2.1. Under the assumptions in Lemma 2.1, consider any $t \in \mathbb{R}$, and any $w \in V(t)$. Then, it results that

$$\big(\big(P_k(t)v, w\big)\big) = 0, \qquad \forall v \in V, \text{ and } \forall k \geq 1. \tag{2.21}$$

In fact, $P_k(t)v$ belongs to $V(t)^{\perp}$: this follows from (2.15), since the family $\{V(t)^{\perp}\}$ is non-increasing, and $\operatorname{supp}(\varphi_k(t)) \subset [-1/k, 0]$ (see (2.14)).

In the sequel, we will also use the following lemma (for its proof we refer to [6]).

LEMMA 2.2. *Under the assumptions in Lemma* 2.1, *take any* $u(t) \in L^2(0, T; V)$. *Then, it results that*

$$P_k(t)u(t) \to P(t)u(t) \qquad \text{strongly in } L^2(0, T; V), \text{ as } k \to +\infty. \tag{2.22}$$

2.3. For our proofs in the following sections, we also need to approximate the given functions "$t \to A(t)$" (see (2.4), (2.5), (2.6)), and "$t \to f(t)$" (see (2.7)) by more regular maps. We do it in the present subsection, by also using the convolution method, and proceeding as Arosio did in [2]. Let us introduce the following notation:

$$\begin{aligned}&\text{if } X \text{ is a Banach space}, a, b \in [-\infty, +\infty], \text{ and } g(t) \in BV(a, b; X),\\ &\mathsf{V}\,(a, b; g(t); X) \text{ denotes the } \textit{total variation} \text{ of } g(t) \text{ in}]a, b[.\end{aligned} \tag{2.23}$$

Consider now the BV-functions "$t \to A(t)$" and "$t \to f(t)$". The left (resp. right) limits $A(t^-)$, $f(t^-)$ (resp. $A(t^+), f(t^+)$) exist for every $t \in]0, T]$ (resp. $t \in [0, T[$). Then, denote by $\tilde{A}(t)$ (resp. $\tilde{f}(t)$) the extension of $A(t)$ (resp. $f(t)$) to all of $\mathbb{R}$, such that $\tilde{A}(t) = A(0^+)$ (resp. $\tilde{f}(t) = f(0^+)$) for every $t < 0$, and such that $\tilde{A}(t) = A(T^-)$ (resp. $\tilde{f}(t) = f(T^-)$) for every $t > T$. Consider now a sequence $\{\varphi_k(t)\}_{k \geq 1}$ satisfying (2.14), and define $(\forall k \geq 1, \forall t \in \mathbb{R})$

$$\chi_k(t) \equiv \varphi_k(-t). \tag{2.24}$$

Then, we define $(\forall k \geq 1, \forall t \in \mathbb{R})$

$$A_k(t) \equiv \tilde{A}(t) * \chi_k(t); \qquad f_k(t) \equiv \tilde{f}(t) * \chi_k(t), \tag{2.25}$$

where the convolution product $*$ is meant as in (2.15). It is clear that $(\forall k \geq 1)$

$$\text{"}t \to A_k(t)\text{"} \in C^\infty(\mathbb{R}; \mathsf{L}\,(V, V^*)); \qquad \text{"}t \to f_k(t)\text{"} \in C^\infty(\mathbb{R}; V^*). \tag{2.26}$$

We list now various properties of such functions: some of them are obvious; the others are proved in [2]. First, consider the functions "$t \to A_k(t)$." Thanks to (2.4), (2.5), (2.6), (2.24), (2.25), it results that ($\forall k \geq 1$)

$$(A_k(t)u, v) = \overline{(A_k(t)v, u)}, \qquad \forall u, v \in V, \forall t \in \mathbb{R}; \tag{2.27}$$

$$(A_k(t)u, u) + \lambda|u|^2 \geq c\|u\|^2, \qquad \forall u \in V, \forall t \in \mathbb{R}; \tag{2.28}$$

$$A_k(0) = A(0^+). \tag{2.29}$$

Moreover, by denoting by $|||\cdot|||$ the usual operator norm in $\mathcal{L}(V, V^*)$, and using the notation (2.23), we also have that

$$\lim_{k \to +\infty} |||A_k(t) - A(t)||| = 0, \qquad \text{for a.a. } t \in]0, T[; \tag{2.30}$$

$$|||A_k(t)||| \leq \operatorname*{ess\,sup}_{\tau \in]0,T[} |||A(\tau)||| \equiv M, \qquad \forall t \in \mathbb{R}, \forall k \geq 1, \tag{2.31}$$

and hence, in particular,

$$A_k(t) \to A(t) \qquad \text{strongly in } L^1(0, T; \mathcal{L}(V, V^*)), \text{ as } k \to +\infty; \tag{2.32}$$

$$\begin{aligned} \int_0^t |||A'_k(\tau)|||\, d\tau \\ &= \mathcal{V}(0, t; A_k(\tau); \mathcal{L}(V, V^*)) \\ &\leq \mathcal{V}(0, t; A(\tau); \mathcal{L}(V, V^*)), \qquad \forall t \in [0, T], \forall k \geq 1. \end{aligned} \tag{2.33}$$

Next, let us consider the functions "$t \to f_k(t)$." Thanks to (2.7), (2.24), and (2.25), it results that

$$f_k(0) = f(0^+), \qquad \forall k \geq 1; \tag{2.34}$$

$$\lim_{k \to +\infty} \|f_k(t) - f(t)\|_* = 0, \qquad \text{for a.a. } t \in]0, T[; \tag{2.35}$$

$$\|f_k(t)\|_* \leq \operatorname*{ess\,sup}_{\tau \in]0,T[} \|f(\tau)\|_* \equiv N, \qquad \forall t \in \mathbb{R}, \forall k \geq 1; \tag{2.36}$$

$$\begin{aligned} \int_0^t \|f'_k(\tau)\|_*\, d\tau &= \mathcal{V}(0, t; f_k(\tau); V^*) \\ &\leq \mathcal{V}(0, t; f(\tau); V^*), \qquad \forall t \in [0, T], \forall k \geq 1. \end{aligned} \tag{2.37}$$

We now state another lemma, which will be used in Section 3. First, let us introduce the following notation. Since (2.1) holds,

we denote by J the canonical antiduality operator from V into V^*,

$$\text{which is defined by } (Ju, v) = ((u, v)), \forall u, v \in V. \tag{2.38}$$

It is obvious that J is an isometric isomorphism of V onto V^*.

LEMMA 2.3. *Let* (2.1), (2.2), (2.11), *and* (2.12) *hold, and consider the definitions* (2.13), (2.14), *and* (2.15). *Let* $A(t)$ (*resp.* $f(t)$) *satisfy* (2.4), (2.5), *and* (2.6) (*resp.* (2.7)), *and consider* $A(0^+)$, *and* $f(0^+)$. *Take any* $u_0 \in V(0)$. *Then, there exists a sequence* $\{u_{0k}\}_{k \geq 1}$ *such that*

$$\begin{aligned}
&\text{(a)} \quad u_{0k} \in V, \forall k \geq 1;\\
&\text{(b)} \quad u_{0k} \to u_0 \text{ strongly in } V, \text{ as } k \to +\infty;\\
&\text{(c)} \quad [f(0^+) - iA(0^+)u_{0k} - ikJP_k(0)u_{0k}] \in V, \forall k \geq 1;\\
&\text{(d)} \quad k\|P_k(0)u_{0k}\| \leq c^{-1}, \forall k \geq 1 \ (\text{where } c \text{ is given in } (2.6)).
\end{aligned} \tag{2.39}$$

The proof of this lemma is similar to the one of [6, Lemma 2.3].

3. EXISTENCE OF WEAK SOLUTIONS

We prove, in this section, that a weak solution to (1.1)–(1.2)–(1.3) *actually exists*, when we assume that (besides the other "natural" hypotheses) *the monotonicity condition* (2.11) *holds*.

THEOREM 3.1. *Let* (2.1), (2.2), (2.11), *and* (2.4), (2.5), (2.6) *hold. Take any* $f(t)$ *as in* (2.7), *and any* $u_0 \in V(0)$. *Then, there exists a* (*not necessarily unique*) *weak solution* $u(t)$ *to the problem* (1.1)–(1.2)–(1.3) (*i.e., a function* $u(t)$ *satisfying* (2.9) *and* (2.10)).

Proof. Our proof consists of the following steps: (a) approximation by means of penalization and regularization, (b) estimates, (c) passage to the limit.

(a) First, consider the definition (2.15) of $P_k(t)$, and the definitions (2.25) of $A_k(t)$, and of $f_k(t)$. Now, since $u_0 \in V(0)$, Lemma 2.3 applies; so, let $\{u_{0k}\}_{k \geq 1}$ be a sequence satisfying (2.39). Then, take *any* integer $k \geq 1$, and consider the problem

$$u'_k(t) + iA_k(t)u_k(t) + ikJP_k(t)u_k(t) = f_k(t), \qquad 0 < t < T; \tag{3.1}$$

$$u_k(0) = u_{0k}. \tag{3.2}$$

Now, we have to recall that $A_k(t)$ satisfies (2.26), (2.27), (2.28), (2.29); J is defined in (2.38); $P_k(t)$ satisfies (2.16), (2.17), (2.18); $f_k(t)$ satisfies (2.26), (2.34); u_{0k} satisfies (2.39) (in particular, (2.39)(a) and (c)). Then, thanks to these properties, starting from a result by Pozzi (see [21, Teor. 4.1 and Oss. 4.6]), we can use, e.g., a differential quotients argument to obtain that

$$\begin{aligned}
&\text{there exists a unique } u_k(t) \in C^1([0,T];V) \cap C^2([0,T];V^*),\\
&\text{solution to } (3.1)-(3.2).
\end{aligned} \tag{3.3}$$

(b) Since $A_k(t) + kJP_k(t)$ is only *weakly* V-coercive (see (2.28), (2.18), and (2.38)), it is more convenient (for the following estimates) to perform the following change of unknown function,

$$v_k(t) = e^{-i\lambda t} u_k(t), \qquad \forall t \in [0, T]; \forall k \geq 1; \tag{3.4}$$

hence, $v_k(t)$ satisfies

$$\begin{aligned} v_k'(t) + iA_k(t) v_k(t) + i\lambda v_k(t) + ikJP_k(t) v_k(t) \\ = e^{-i\lambda t} f_k(t), \qquad 0 < t < T; \end{aligned} \tag{3.5}$$

$$v_k(0) = u_{0k}. \tag{3.6}$$

Clearly, $v_k(t)$ (unique solution to (3.5)–(3.6)) has the same regularity properties of $u_k(t)$. Observe, moreover, that $A_k(t) + \lambda I + kJP_k(t)$ is *strongly* V-coercive. (So, remark that one could use here a result of [8] (concerning, more generally, variational inequalities; take [8, Theorem 2.1] with $K = V$) to establish *directly* the existence and the uniqueness of the solution $v_k(t)$ to the problem (3.5)–(3.6) in the appropriate functional framework.) Now, consider *any* integer $k \geq 1$, and take $v_k(t) \in C^1([0, T]; V) \cap C^2([0, T]; V^*)$, solution to (3.5)–(3.6); we "multiply" (in the antiduality pairing between V^* and V) both sides of (3.5) by $v_k'(t)$. By taking the imaginary parts, and using (2.26), (2.27), (2.38), (2.16), (2.17), we get

$$\begin{aligned} &\frac{d}{dt}\big(A_k(t) v_k(t), v_k(t)\big) - \big(A_k'(t) v_k(t), v_k(t)\big) + \lambda \frac{d}{dt} |v_k(t)|^2 \\ &\quad + k\frac{d}{dt}\big(\big(P_k(t) v_k(t), v_k(t)\big)\big) - k\big(\big(P_k'(t) v_k(t), v_k(t)\big)\big) \\ &= 2\,\mathrm{Im}\left[\frac{d}{dt}\big(e^{-i\lambda t} f_k(t), v_k(t)\big) - e^{-i\lambda t}\big(f_k'(t) - \lambda f_k(t), v_k(t)\big)\right], \\ &\qquad \forall t \in]0, T[. \end{aligned} \tag{3.7}$$

Next, we take into account (2.20), and we integrate (3.7) from 0 to t $(0 \leq t \leq T)$. By also using (2.29), (2.34), and (3.6), we obtain

$$\begin{aligned} &\big(A_k(t) v_k(t), v_k(t)\big) + k\big(\big(P_k(t) v_k(t), v_k(t)\big)\big) + \lambda |v_k(t)|^2 \\ &\leq \big(A(0^+) u_{0k}, u_{0k}\big) + k\big(\big(P_k(0) u_{0k}, u_{0k}\big)\big) \\ &\quad + \int_0^t \big(A_k'(s) v_k(s), v_k(s)\big)\, ds \\ &\quad + \lambda |u_{0k}|^2 + 2\big|e^{-i\lambda t}\big(f_k(t), v_k(t)\big)\big| + 2\big|\big(f(0^+), u_{0k}\big)\big| \\ &\quad + 2\int_0^t \big|e^{-i\lambda s}\big(f_k'(s) - \lambda f_k(s), v_k(s)\big)\big|\, ds, \qquad \forall t \in [0, T]. \end{aligned} \tag{3.8}$$

Now, it is obvious (from (2.39)(a) and (b)), that there exists a positive number $M_1 = M_1(u_0)$, such that

$$\|u_{0k}\| \le M_1, \qquad \forall k \ge 1. \tag{3.9}$$

Then, starting from (3.8), denoting by $\tilde{c}$ a positive number such that $|v| \le \tilde{c}\|v\|$, $\forall v \in V$ (see (2.1)), and using (2.28), (2.18), (3.9), (2.31), (2.39)(d), (2.36), and some standard inequalities, we get

$$\begin{aligned} c\|v_k(t)\|^2 &+ k\|P_k(t)u_k(t)\|^2 \\ &\le MM_1^2 + M_1c^{-1} + \int_0^t ||| A_k'(s) ||| \cdot \|v_k(s)\|^2\,ds \\ &\quad + \tilde{c}^2\lambda M_1^2 + \frac{2}{c}N^2 + \frac{c}{2}\|v_k(t)\|^2 + 2NM_1 \\ &\quad + 2\int_0^t [\lambda\|f_k(s)\|_* + \|f_k'(s)\|_*] \\ &\quad \cdot\|v_k(s)\|\,ds, \qquad \forall t \in [0,T]. \end{aligned} \tag{3.10}$$

Hence, we obtain

$$\begin{aligned} \frac{c}{2}\|v_k(t)\|^2 \le MM_1^2 &+ M_1c^{-1} + \lambda\tilde{c}^2M_1^2 + \frac{2}{c}N^2 + 2NM_1 \\ &+ \int_0^t ||| A_k'(s) ||| \cdot \|v_k(s)\|^2\,ds \\ &+ 2\int_0^t [\lambda\|f_k(s)\|_* + \|f_k'(s)\|_*] \\ &\cdot\|v_k(s)\|\,ds, \qquad \forall t \in [0,T]. \end{aligned} \tag{3.11}$$

Then, we use here a suitable generalized Gronwall lemma (see, e.g., Baiocchi [3]), and we get

$$\begin{aligned} \|v_k(t)\| \le 2\Bigg[\sqrt{\frac{2}{c}}\left(MM_1^2 + M_1c^{-1} + \lambda\tilde{c}^2M_1^2 + \frac{2}{c}N^2 + 2NM_1\right)^{1/2} \\ + \frac{4}{c}\int_0^T (\lambda\|f_k(s)\|_* + \|f_k'(s)\|_*)\,ds\Bigg] \\ \cdot\exp\left(\frac{4}{c}\int_0^T ||| A_k'(s) |||\,ds\right), \qquad \forall t \in [0,T]. \end{aligned} \tag{3.12}$$

We take into account (2.33), (2.36), and (2.37). So, from (3.12), we obtain that

$$\text{there exists a positive number } C, \text{ depending on } T, V, H, A(t), f(t), u_0, \\ \textit{but independent of } k \textit{ and of } t, \text{ such that } \|v_k(t)\| \le C, \forall t \in [0,T], \forall k \ge 1. \tag{3.13}$$

Now, we come back to (3.10), and we take into account (3.13), (2.33), (2.36), and (2.37). Then, we also get that

$$\text{there exists a positive number } D, \textit{ independent of } k \textit{ and of } t, \\ \text{such that } k^{1/2}\|P_k(t)v_k(t)\| \le D, \forall t \in [0,T], \forall k \ge 1. \tag{3.14}$$

Hence, thanks to (3.4), we have actually obtained that

$$\{u_k(t)\}_{k\ge 1} \text{ is bounded in } L^\infty(0,T;V); \tag{3.15}$$

$$\{k^{1/2}P_k(t)u_k(t)\}_{k\ge 1} \text{ is bounded in } L^\infty(0,T;V). \tag{3.16}$$

(c) First, it is obvious (from (3.16)) that

$$P_k(t)u_k(t) \to 0 \qquad \text{strongly in } L^\infty(0,T;V), \text{ as } k \to +\infty. \tag{3.17}$$

Moreover, thanks to (3.15), we can extract from $\{u_k(t)\}_{k\ge 1}$ *a subsequence, still denoted by* $\{u_k(t)\}_{k\ge 1}$, such that, as $k \to +\infty$,

$$u_k(t) \rightharpoonup u(t) \qquad \text{weakly star in } L^\infty(0,T;V), \\ \text{and also weakly in } L^2(0,T;V). \tag{3.18}$$

We will show that such $u(t)$ satisfies, in fact, (2.9) and (2.10). First, to verify that (2.9) holds, we need only to prove that

$$u(t) \in V(t) \qquad \text{for a.a. } t \in]0,T[. \tag{3.19}$$

Toward this aim, consider the above subsequence $\{u_k(t)\}_{k\ge 1}$, and take *any* $v(t) \in L^2(0,T;V)$. By using (2.17), (3.18), and Lemma 2.2, we get that

$$\begin{aligned} &\int_0^T ((P(t)u(t) - P_k(t)u_k(t), v(t)))\,dt \\ &\quad = \int_0^T ((P(t)u(t) - P_k(t)u(t), v(t)))\,dt \\ &\qquad + \int_0^T ((u(t) - u_k(t), P_k(t)v(t)))\,dt \to 0, \\ &\qquad\qquad \text{as } k \to +\infty, \end{aligned} \tag{3.20}$$

i.e., that

$$P_k(t)u_k(t) \rightharpoonup P(t)u(t) \qquad \text{weakly in } L^2(0,T;V), \text{ as } k \to +\infty. \tag{3.21}$$

Hence, from (3.17) and (3.21), we deduce that $P(t)u(t) = 0$ for a.a. $t \in]0,T[$. Then, thanks also to the definition (2.13), (3.19) is proved.

Now, we show that $u(t)$ satisfies (2.10). Toward this aim, take *any* $w(t) \in W$ (where W is defined in (2.8)), and "multiply" (in the antiduality pairing between V^* and V) both sides of (3.1) by $w(t)$. Thanks to Remark 2.1, we get

$$\begin{aligned}(u_k'(t), w(t)) &+ i(A_k(t)u_k(t), w(t)) \\ &= (f_k(t), w(t)), \qquad \text{for a.a. } t \in]0,T[.\end{aligned} \tag{3.22}$$

Next, integrate (3.22) from 0 to T. Then, by making an integration by parts, and using (2.8) and (3.2), we obtain

$$\begin{aligned} i\int_0^T (A_k(t)u_k(t), w(t))\,dt &- \int_0^T \overline{(w'(t), u_k(t))}\,dt \\ &= \overline{(w(0), u_{0k})} + \int_0^T (f_k(t), w(t))\,dt. \end{aligned} \tag{3.23}$$

Take now, in (3.23), *any element* $u_k(t)$ *of the above subsequence* $\{u_k(t)\}_{k \geq 1}$, *satisfying* (3.18). Thanks to (2.8), (3.18), (2.39)(b), it is clear that, as $k \to +\infty$,

$$\begin{gathered} \overline{(w(0), u_{0k})} \to \overline{(w(0), u_0)}; \\ \int_0^T \overline{(w'(t), u_k(t))}\,dt \to \int_0^T \overline{(w'(t), u(t))}\,dt. \end{gathered} \tag{3.24}$$

On the other hand, we claim that, as $k \to +\infty$,

$$\begin{aligned} &\text{(a)} \qquad \int_0^T (A_k(t)u_k(t), w(t))\,dt \to \int_0^T (A(t)u(t), w(t))\,dt; \\ &\text{(b)} \qquad \int_0^T (f_k(t), w(t))\,dt \to \int_0^T (f(t), w(t))\,dt. \end{aligned} \tag{3.25}$$

(Remark that in (3.24) we used the properties of $w(t)$ as a V^*-valued function, while we need $w(t) \in L^1(0,T;V)$ to obtain (3.25).) To verify (3.25)(a), we use first (2.27), and we take into account (3.18). Then (3.25)(a) is proved, if we are able to show that

$$A_k(t)w(t) \to A(t)w(t) \qquad \text{strongly in } L^1(0,T;V^*), \text{ as } k \to +\infty. \tag{3.26}$$

Now, (3.26) is, in fact, true, thanks to (2.30), (2.31), and to the Lebesgue dominated convergence theorem. Similarly, (3.25)(b) can also be proved, by using (2.35), (2.36), and the Lebesgue theorem again.

Hence, from (3.23), (3.24), and (3.25), we get that $u(t)$ satisfies (2.10). So, Theorem 3.1 is completely proved.

For the sequel, we also need the following notation:

$$\text{if } X \text{ is any Banach space, then } X_w \text{ denotes the space } X \text{ endowed with its weak topology.} \tag{3.27}$$

THEOREM 3.2. *Let all of the assumptions of Theorem* 3.1 *hold. Suppose, moreover, that*

$$V(0) \textit{ is dense in } H \tag{3.28}$$

(*and hence, thanks to* (2.11), *any* $V(t)$ $(0 \leq t \leq T)$ *is dense in* H). *Then, every weak solution* $u(t)$ *to* (1.1)–(1.2)–(1.3), *obtained through Theorem* 3.1, *satisfies moreover* $u(t) \in C^0([0,T]; V_w)$ (*and in this sense* (1.3), *i.e.*, $u(0) = u_0$, *can also be meant*).

Proof. We start as we did in part (a) of the proof of Theorem 3.1. So, let us consider the sequences $\{P_k(t)\}$, $\{A_k(t)\}$, $\{f_k(t)\}$, $\{u_{0k}\}$. Let us also consider the problems (3.1)–(3.2), and the sequence $\{u_k(t)\}_{k \geq 1}$ of the corresponding solutions $u_k(t) \in C^1([0,T]; V) \cap C^2([0,T]; V^*)$ (see (3.3)). Let $\{u_k(t)\}_{k \geq 1}$ also denote a subsequence satisfying (3.18) (where, as we know, $u(t)$ fulfils (2.9) and (2.10)). We will show that

$$\forall h \in H, \text{ the sequence } \{(u_k(t), h)\}_{k \geq 1} \text{ is equicontinuous on } [0,T]. \tag{3.29}$$

Thus, since also (3.4) and (3.13) hold, we can use the Ascoli–Arzelà theorem, to deduce that there exists a subsequence, still denoted by $\{(u_k(t), h)\}_{k \geq 1}$, which converges (to $(u(t), h)$, of course) in $C^0([0,T])$, as $k \to +\infty$. Hence, we have that $u(t) \in C^0([0,T]; H_w)$. On the other hand, $u(t)$ satisfies, in particular, $u(t) \in L^\infty(0,T; V)$ (see (2.9)). Hence, thanks also to (2.1), we can use Lemma 8.1 of Lions and Magenes [19, Chap. 3] to obtain that $u(t) \in C^0([0,T]; V_w)$. So, by using also (3.2), (2.39)(a), and (2.39)(b), it is clear that in this sense $u(0) = u_0$ can also be meant. Then, the present theorem is proved, if we verify that (3.29) holds. Toward this aim, take *any* $v \in V(0)$, and "multiply" by v (in the antiduality pairing between V^* and V) both sides of (3.1). Then, since $P_k(t)u_k(t) \in V(t)^\perp \subseteq V(0)^\perp$ (see Remark 2.1), we get

$$\frac{d}{dt}(u_k(t), v) + i(A_k(t)u_k(t), v) = (f_k(t), v), \qquad \forall t \in]0,T[. \tag{3.30}$$

Next, we integrate (3.30) from t_1 to t_2 $(0 \le t_1 \le t_2 \le T)$, and we take into account (2.31), (2.36), (3.4), and (3.13). Hence, we obtain

$$\begin{aligned}
&|(u_k(t_2) - u_k(t_1), v)| \\
&\quad \le \left| \int_{t_1}^{t_2} (A_k(t) u_k(t), v)\, dt \right| + \left| \int_{t_1}^{t_2} (f_k(t), v)\, dt \right| \\
&\quad \le \|v\| \cdot \left[\int_{t_1}^{t_2} |||(A_k(t)||| \cdot \|u_k(t)\|\, dt + \int_{t_1}^{t_2} \|f_k(t)\|_*\, dt \right] \\
&\quad \le \|v\| (MC + N)(t_2 - t_1), \qquad \forall k \ge 1. \qquad (3.31)
\end{aligned}$$

Take now *any* $h \in H$. Then, by using (3.4), (3.13), and (3.31), we get, *for every* $v \in V(0)$, *and for any* t_1 *and* t_2 *with* $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned}
&|(u_k(t_2) - u_k(t_1), h)| \\
&\quad \le |(u_k(t_2) - u_k(t_1), h - v)| + |(u_k(t_2) - u_k(t_1), v)| \\
&\quad \le 2C|h - v| + \|v\| (MC + N)(t_2 - t_1), \qquad \forall k \ge 1. \quad (3.32)
\end{aligned}$$

Hence, thanks to (3.32) and to assumption (3.28), it is clear that (3.29) holds.

4. UNIQUENESS OF THE WEAK SOLUTION

We prove, in this section, our *uniqueness* result, by assuming that (besides the other "natural" hypotheses) $\{V(t)\}$ *is a non-increasing family* (see (4.1) below). Since our problem is a *linear* problem, the uniqueness of the weak solution to (1.1)–(1.2)–(1.3), is given by the following theorem.

THEOREM 4.1. *Let* (2.1), (2.2), (2.4), (2.5), *and* (2.6) *hold. Assume, moreover, that*

$$V(t_1) \supseteq V(t_2), \qquad \forall t_1, t_2 \text{ such that } 0 \le t_1 \le t_2 \le T. \qquad (4.1)$$

Let $u(t)$ *satisfy* (2.9) *and* (2.10), *with* $u_0 = 0$ *and* $f(t) = 0$, *for a.a.* $t \in]0, T[$. *Then, it results that* $u(t) = 0$, *for a.a.* $t \in]0, T[$.

Proof. We prove our uniqueness result, by suitably modifying an argument, due to Ladyzenskaja [16] (also see, e.g., Lions [17], Lions and Magenes [19]). So, let us fix *any* $s \in]0, T]$, and define

$$v(t) = -e^{i\lambda t} \int_t^s e^{-i\lambda\tau} u(\tau)\, d\tau,$$

$$\forall t \in [0, s];\ v(t) = 0, \forall t \in [s, T], \text{ where } \lambda \text{ is given as in } (2.6). \qquad (4.2)$$

Since $u(t)$ satisfies (2.9), and (4.1) holds, we have that $v(t)$ is an "admissible test-function," i.e., that $v(t) \in W$ (where W is defined in (2.8)). In fact, we have something better, i.e., that $v(t) \in C^0([0,T];V)$, $v(t) \in V(t)$, $\forall t \in [0,T]$, $v'(t) \in L^\infty(0,T;V(t))$, along with $v(T) = 0$.

Now, by taking $w(t) = v(t)$ in (2.10) (where $u_0 = 0$ and $f(t) \equiv 0$), we get

$$i\int_0^s \left[(A(t)v'(t), v(t)) + \lambda\overline{(v'(t), v(t))} \right] dt$$

$$= \int_0^s \left[|v'(t)|^2 - \lambda(A(t)v(t), v(t)) \right] dt. \tag{4.3}$$

So, by considering the imaginary parts, we find

$$\operatorname{Re} \int_0^s ([A(t) + \lambda I]v'(t), v(t))\, dt$$

$$= 0, \qquad \text{where } I \text{ denotes the identity operator.} \tag{4.4}$$

Next, go back to Subsection 2.3, and recall the definition (2.25) of $A_k(t)$ ($k \geq 1, t \in \mathbb{R}$), along with the properties (2.26) and (2.27). Then, from (4.4), we obtain, for every integer $k \geq 1$,

$$\begin{aligned} 0 = {} & \operatorname{Re} \int_0^s ([A_k(t) + \lambda I]v'(t), v(t))\, dt \\ & + \operatorname{Re} \int_0^s ([A(t) - A_k(t)]v'(t), v(t))\, dt \\ = {} & \frac{1}{2}([A_k(s) + \lambda I]v(s), v(s)) - \frac{1}{2}([A_k(0) + \lambda I]v(0), v(0)) \\ & - \frac{1}{2}\int_0^s (A_k'(t)v(t), v(t))\, dt \\ & + \operatorname{Re} \int_0^s ([A(t) - A_k(t)]v'(t), v(t))\, dt. \end{aligned} \tag{4.5}$$

Now, by using (4.2), (2.29), (2.33) (along with the notation (2.23)), we get from (4.5), for every integer $k \geq 1$,

$$\begin{aligned} & (A(0^+)v(0), v(0)) + \lambda|v(0)|^2 \\ & \quad \leq \Big(\sup_{t\in[0,s]} \|v(t)\| \Big)^2 \cdot \mathcal{V}(0, s; A(t); \mathcal{L}(V, V^*)) \\ & \qquad + 2\Big(\operatorname{ess}\sup_{t\in\,]0,s[} \|v'(t)\| \Big) \cdot \Big(\sup_{t\in[0,s]} \|v(t)\| \Big) \\ & \qquad \cdot \int_0^s ||| A(t) - A_k(t) |||\, dt. \end{aligned} \tag{4.6}$$

Next, we pass to the limit as $k \to +\infty$, and we take into account (2.32). By also using (2.6), we obtain

$$c\|v(0)\|^2 \leq \left(\sup_{t\in[0,s]} \|v(t)\| \right)^2 \cdot \mathsf{V}(0, s; A(t); \mathsf{L}(V, V^*)). \qquad (4.7)$$

(Remark that, when $A(t)$ is "more regular" than in (2.4) (e.g., when $A(t) \in W^{1,1}(0, T; \mathsf{L}(V, V^*))$), (4.7) can be deduced directly from (4.4), (2.5), (2.6), without using the approximation through the "smooth" operators $A_k(t)$. In the general case (i.e., when (2.4) holds), the above procedure comes from [2].) Now, we define

$$z(t) = \int_0^t e^{-i\lambda\tau} u(\tau)\, d\tau, \qquad \forall t \in [0, T], \qquad (4.8)$$

and we observe that (thanks also to (4.2))

$$v(t) = e^{i\lambda t}(z(t) - z(s)),$$
$$\forall t \in [0, s], \text{ and, in particular, } v(0) = -z(s). \qquad (4.9)$$

Then, we put (4.9) in (4.7). So, by considering that an *arbitrary* $s \in]0, T]$ was taken, and that $z(0) = 0$, we get

$$c\|z(s)\|^2 \leq 4\left(\sup_{t\in[0,s]} \|z(t)\|^2 \right)$$
$$\cdot \mathsf{V}(0, s; A(t); \mathsf{L}(V, V^*)), \quad \forall s \in [0, T]. \qquad (4.10)$$

Next, thanks to (2.4), *there exists* $\delta \in]0, T]$, *such that* $\mathsf{V}(0, \delta; A(t); \mathsf{L}(V, V^*)) < c/8$ (for example). Hence we obtain

$$c\|z(s)\|^2 \leq \frac{c}{2}\left(\sup_{t\in[0,s]} \|z(t)\|^2 \right), \qquad \forall s \in [0, \delta]. \qquad (4.11)$$

Now, from (4.11), we can readily deduce that $z(t) = 0$, $\forall t \in [0, \delta]$ and, hence, thanks to (4.8), $u(t) = 0$, for a.a. $t \in]0, \delta[$. Let us prove that, *in fact*, $u(t) = 0$, for a.a. $t \in]0, T[$. By assuming that $\delta < T$, and proceeding, e.g., as in [2], we define $\gamma \equiv \sup\{t \in]0, T] \mid u(s) = 0, \text{ for a.a. } s \in]0, t[\}$; hence $\gamma \geq \delta$. Suppose that $\gamma < T$. Then, we can use the above procedure (where γ is taken in place of 0), and obtain that $u(t) = 0$ almost everywhere in a right neighbourhood of γ. Then, we have a contradiction. So, Theorem 4.1 is completely proved.

5. SOME EXAMPLES AND REMARKS

5.1. If we consider Theorems 3.1 and 4.1 *together*, we see that we can obtain the existence *and* the uniqueness of the weak solution $u(t)$ to (1.1)–(1.2)–(1.3), when (besides the other assumptions) we have that $V(t) = \tilde{V}$, $\forall t \in [0, T]$, where $\tilde{V}$ is a (fixed) closed subspace of V. If, in particular, $\tilde{V} = V$, we deduce from Definition 2.1 that $u(t)$ satisfies (1.2) in the sense (e.g.) of $\mathcal{D}'(0, T; V^*)$ (and that $u'(t) \in L^\infty(0, T; V^*)$). So, in this case, Theorems 3.1, 3.2, and 4.1 give the following result.

COROLLARY 5.1. *Let* (2.1), (2.4), (2.5), *and* (2.6) *hold*. *Then*, *for any* $f(t)$ *as in* (2.7), *and any* $u_0 \in V$, *there exists a unique* $u(t) \in C^0([0, T]; V_w) \cap W^{1,\infty}(0, T; V^*)$, *satisfying* (1.2) (*in the sense* (*e.g.*) *of* $\mathcal{D}'(0, T; V^*)$), *and* (1.3).

Remark 5.1. Let us observe that, by using a suitable procedure (which is similar to the one employed in [2, Theorem 1.1] for differential equations of hyperbolic type), we could *also* obtain that, in Corollary 5.1, $u(t)$ belongs to $C^0([0, T]; V)$ (and not only to $C^0([0, T]; V_w)$). Moreover, we remark that the result of Corollary 5.1 gives something *new*, as far as we know about abstract differential equations of Schroedinger-type. In fact, in Corollary 5.1 we assume that "$t \to A(t)$" and "$t \to f(t)$" are *BV-functions* on $]0, T[$, while the previous results on (1.2)–(1.3) require that "$t \to A(t)$" and "$t \to f(t)$" have to be *at least* $W^{1,1}$-functions on $]0, T[$. (See, e.g., Lions [17], Lions and Magenes [19], and, *in particular*, Pozzi [21].)

5.2. Let $T > 0$ be given. Let Ω be an open bounded subset of $\mathbb{R}^N$, whose boundary $\Gamma \equiv \partial\Omega$ is a $(N-1)$-dimensional manifold of class (e.g.) C^1. Let Q denote the open cylinder $Q \equiv \Omega \times]0, T[$, and $\Sigma \equiv \Gamma \times]0, T[$ the lateral boundary of Q. Let moreover $\{\Gamma_0(t)\}_{t \in [0,T]}$ be a family of C^1-submanifolds (with boundary) of Γ, and define

$$\Sigma_0 \equiv \bigcup_{0<t<T} \Gamma_0(t) \times \{t\}; \qquad \Sigma_1 \equiv \Sigma \setminus \overline{\Sigma}_0. \tag{5.1}$$

Let us consider the following second order linear differential operator $\mathcal{A}$,

$$\mathcal{A}u \equiv -\sum_{k,j=1}^{N} \frac{\partial}{\partial x_k}\left(a_{kj}(x,t) \frac{\partial u}{\partial x_j} \right) + c(x,t)u, \tag{5.2}$$

where $a_{kj}(x, t)$ $(k, j = 1, \ldots, N)$ and $c(x, t)$ are given (complex-valued) functions (defined in $\overline{Q}$). Let $\nu_{\mathcal{A}} = \nu_{\mathcal{A}}(x, t)$ be the related conormal vector to Σ. Then, we consider, *in a formal way*, the following Cauchy-mixed problem. Given $f(x,t)$, and $u_0(x)$, to find $u(x,t)$, such that

$$\begin{aligned}
&\text{(a)} \qquad \frac{\partial u}{\partial t}(x,t) + iAu(x,t) = f(x,t) \text{ in } Q;\\
&\text{(b)} \qquad u(x,t) = 0 \text{ on } \Sigma_0;\\
&\text{(c)} \qquad \frac{\partial u}{\partial \nu_A}(x,t) = 0 \text{ on } \Sigma_1;\\
&\text{(d)} \qquad u(x,0) = u_0(x) \text{ in } \Omega.
\end{aligned} \tag{5.3}$$

Let us consider the weak solutions to the problem (5.3), according to Definition 2.1. Toward this aim, we take

$$H = L^2(\Omega), \qquad V = H^1(\Omega) \ \big(\text{and hence } V^* = (H^1(\Omega))^*\big), \tag{5.4}$$

so that (2.1) holds. Moreover, we take, for every $t \in [0,T]$,

$$\begin{aligned}
V(t) &= H^1_{\Gamma_0(t)}(\Omega)\\
&\equiv \big\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0(t) \text{ (in the sense of traces)} \big\},
\end{aligned} \tag{5.5}$$

which is a closed subspace of V, and is *dense* in H. We have that

$$\begin{aligned}
&\text{(2.11) (resp. (4.1)) holds iff } \{\Gamma_0(t)\}_{t\in[0,T]} \text{ is}\\
&\text{a non-increasing (resp. non-decreasing) family with } t.
\end{aligned} \tag{5.6}$$

Now, let us consider the operator A. We assume that:

$$\begin{aligned}
&\text{(a)} \qquad \text{“}t \to a_{kj}(\cdot,t)\text{” } (k,j = 1,\dots,N) \text{ and “}t \to c(\cdot,t)\text{”}\\
&\qquad\qquad \text{belong to the space } BV(0,T; L^\infty(\Omega));\\
&\text{(b)} \qquad a_{kj}(x,t) = \overline{a_{jk}(x,t)} \text{ for a.a. } (x,t) \in Q \ (k,j = 1,\dots,N),\\
&\qquad\qquad \text{and the function } c(x,t) \text{ is real-valued};\\
&\text{(c)} \qquad \exists \alpha > 0, \text{ such that } \sum_{k,j=1}^{N} a_{kj}(x,t)\,\xi_k\,\bar{\xi}_j \geq \alpha \sum_{k=1}^{N} |\xi_k|^2,\\
&\qquad\qquad \forall(\xi_1,\dots,\xi_N) \in \mathbb{C}^N, \text{ and for a.a. } (x,t) \in Q.
\end{aligned} \tag{5.7}$$

Then, we define, for a.a. $t \in]0,T[$, and for any $u,v \in V = H^1(\Omega)$,

$$a(t;u,v) \equiv \sum_{k,j=1}^{N} \int_\Omega a_{kj}(x,t) \frac{\partial u}{\partial x_k}(x) \overline{\frac{\partial v}{\partial x_j}(x)}\, dx$$
$$+ \int_\Omega c(x,t) u(x) \overline{v(x)}\, dx. \tag{5.8}$$

Hence, we have that $\{a(t;\cdot,\cdot) \mid \text{ for a.a. } t \in]0,T[\}$ is a family of sesquilinear and continuous forms on $V \times V$. Then, we can define, for a.a. $t \in]0,T[$, and for any $u,v \in V$,

$$(A(t)u,v) \equiv a(t;u,v), \tag{5.9}$$

where $(\cdot,\cdot)$ denotes the antiduality pairing between V^* and V. We have thus defined a family $\{A(t) \mid \text{ for a.a. } t \in]0,T[\}$ of linear and continuous operators from V into V^* (which are related, in a natural way, to $\mathcal{A}$ and to V). Then, thanks to (5.7), such a family $\{A(t)\}$ satisfies (2.4), (2.5), and (2.6). Take now

$$\begin{aligned} &\text{(a)} \qquad f(x,t), \text{ where } \text{“} t \to f(\cdot,t) \text{”} \in BV\big(0,T;\big(H^1(\Omega)\big)^*\big); \\ &\text{(b)} \qquad u_0(x) \in V(0) = H^1_{\Gamma_0(0)}(\Omega). \end{aligned} \tag{5.10}$$

Now, by means of (5.4), (5.5), and of (5.7)–(5.10), we can use Definition 2.1 to give *a precise notion of a weak solution to* (5.3). For the sake of brevity, we do not rewrite here Definition 2.1 in the present case. Let us only remark that (5.3)(b) is “contained” in (2.9); (5.3)(a), (c), (d) are “contained” in (2.10). Then, by considering (5.6), we see that Theorems 3.1 and 3.2 (resp. Theorem 4.1) apply here, when $\{\Gamma_0(t)\}$ is a non-increasing (resp. non-decreasing) family with t.

5.3. Let $T > 0$ be given. Let $\{\Omega(t)\}_{t \in [0,T]}$ be a family of bounded open subsets $\Omega(t)$ of $\mathbb{R}^N$. We assume that, for every $t \in [0,T]$, the boundary $\Gamma(t) \equiv \partial\Omega(t)$ of $\Omega(t)$ is a $(N-1)$-dimensional manifold of class (e.g.) C^1. Let us define

$$Q \equiv \bigcup_{0<t<T} \Omega(t) \times \{t\}; \qquad \Sigma \equiv \bigcup_{0<t<T} \Gamma(t) \times \{t\}; \qquad B \equiv \mathbb{R}^N \times]0,T[, \tag{5.11}$$

and assume that Q is an *open* subset of $\mathbb{R}^{N+1}$. Let now $\mathcal{A}$ be the differential operator in (5.2). We assume (e.g.) that its coefficients $a_{kj}(x,t)$ $(k,j = 1,\dots,N)$ and $c(x,t)$ are defined in B. Then, we consider, *in a formal way*, the following Cauchy–Dirichlet problem. Given $f(x,t)$, and

$u_0(x)$, to find $u(x,t)$, such that

$$
\begin{aligned}
&\text{(a)} \qquad \frac{\partial u}{\partial t}(x,t) + iAu(x,t) = f(x,t) \qquad \text{in } Q;\\
&\text{(b)} \qquad u(x,t) = 0 \qquad \text{on } \Sigma;\\
&\text{(c)} \qquad u(x,0) = u_0(x) \qquad \text{in } \Omega(0).
\end{aligned}
\tag{5.12}
$$

We now consider the weak solutions to the problem (5.12), according to Definition 2.1. Toward this aim, we take

$$H = L^2(\mathbb{R}^N), \qquad V = H^1(\mathbb{R}^N) \ \big(\text{and hence } V^* = H^{-1}(\mathbb{R}^N)\big), \tag{5.13}$$

so that (2.1) holds. Moreover, we take, for every $t \in [0,T]$,

$$V(t) = \big\{ v \in H^1(\mathbb{R}^N) \mid \operatorname{supp}(v) \subset \overline{\Omega(t)} \big\}, \tag{5.14}$$

which is a closed subspace of V. (Note that, if $v \in V(t)$, then its restriction to $\Omega(t)$ belongs to $H_0^1(\Omega(t))$.) Remark however that $V(t)$ is *not* dense in H. We have that

$$
\begin{aligned}
&(2.11)\ (\text{resp. } (4.1)) \text{ holds here iff } \{\Omega(t)\}_{t\in[0,T]} \text{ is}\\
&\text{a non-decreasing (resp. non-increasing) family with } t.
\end{aligned}
\tag{5.15}
$$

Now, let us consider the operator A (given by (5.2)), and assume that (5.7) holds, where we replace Ω (resp. Q) in (5.7)(a) (resp. (5.7)(b) and (c)) with $\mathbb{R}^N$ (resp. B). Next, define (for a.a. $t \in]0,T[$, and any $u, v \in V = H^1(\mathbb{R}^N)$) $a(t; u, v)$ as in (5.8) (where we replace Ω with $\mathbb{R}^N$). Clearly, $\{a(t;\cdot,\cdot) \mid$ for a.a. $t \in]0,T[\}$ is a family of sesquilinear and continuous forms on $V \times V$. Then, starting from such forms, and proceeding as in (5.9), we can define the related family $\{A(t) \mid$ for a.a. $t \in]0,T[\}$ of linear and continuous operators from $V = H^1(\mathbb{R}^N)$ into $V^* = H^{-1}(\mathbb{R}^N)$. It is clear that, for such a family $\{A(t)\}$, (2.4), (2.5), and (2.6) hold. Take now $u_0(x) \in V(0)$, and "$t \to f(\cdot,t)$" $\in BV(0,T; H^{-1}(\mathbb{R}^N))$. Then, we can use Definition 2.1 to give *a precise notion of a weak solution to* (5.12). For the sake of brevity, we do not rewrite here Definition 2.1 in the present case. Let us only remark that (5.12)(b) is "contained" in (2.9), while (5.12)(a) and (c) are both "contained" in (2.10). Then, by considering (5.15), we see that Theorem 3.1 (resp. Theorem 4.1) applies here, when $\{\Omega(t)\}$ is a non-decreasing (resp. non-increasing) family with t.

Remark 5.2. In this section, the examples concern partial differential operators A of the form (5.2). However, our abstract results also apply to various initial-boundary value problems (either in non-cylindrical regions, or with mixed variable lateral conditions) for *higher order* linear differential operators "of Schroedinger-type" (as, e.g., $\partial/\partial t + i\Delta_x^2$).

Remark 5.3. As far as we know about P.D.E. of Schroedinger-type *in non-cylindrical regions*, we can refer to a note by Medeiros and Milla Miranda [20]: they consider (5.12)(a), with $A = -\Delta_x$ and $f \equiv 0$, in non-cylindrical regions *Q of special type*, and they study the exact boundary controllability. On the other hand, various results are well known about initial-boundary value problems, *in non-cylindrical regions*, for linear (and also nonlinear) P.D.E. of hyperbolic or parabolic type. Considering only the hyperbolic case, we can refer, e.g., to Lions [18], Bardos and Cooper [4], Inoue [14], Sikorav [24], Da Prato and Zolésio [12], and to our paper [6]. We also mention a work by Cannarsa, Da Prato, and Zolésio [9], concerning linear *damped* wave equations in non-cylindrical regions.

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